Group Theory
Week #5, Lecture #19

Group actions on sets (part III)

$g \in G,\ x \in S$

Setup:
- Group G acting on set S: $(g,x) \mapsto g * x = gx$
- $\mathrm{Orb}_G(x)$ — Orbit of x: $Gx := \{gx : g \in G\} \subseteq S$
- $\mathrm{Stab}_G(x)$ — Stabilizer subgroup of x: $G_x := \{g \in G : gx = x\} \leq G$
- $\mathrm{Fix}_G(S)$ — Fixed point set: $S^G := \{x \in S : gx = x,\ \forall g \in G\} \subseteq S$

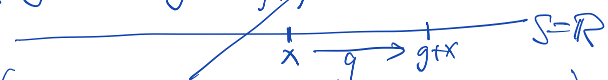
More examples

(1) $G = (\mathbb{R}, +, 0)$ additive group of the real numbers (abelian)

G acting on $S = \mathbb{R}$ via addition:

$$g * x := g + x \qquad g \in G = \mathbb{R},\ x \in S = \mathbb{R}$$

- Orbit of $x \in \mathbb{R}$: $Gx = \{g + x : g \in \mathbb{R}\} = \mathbb{R}$



(since $\forall y \in \mathbb{R}$, $\exists g \in \mathbb{R}$ st $g + x = y$
namely: $g = y - x$)

- Stabilizer subgroups: $G_x = \{g \in \mathbb{R} : g + x = x\} = \{0\}$

$$g + x = x \Leftrightarrow g = 0$$

- Fixed point set: $S^G = \{x \in \mathbb{R} : g + x = x,\ \forall g \in G\} = \emptyset$

$$g + x = x \Leftrightarrow g = 0$$

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(2) **Conjugation action on S_3**

Recall: $S_3 = \{e, a, a^2, b, ab, a^2b\} = D_3$

where $a = (1\,2\,3)$, $b = (1\,2)$ $\left[a: \begin{matrix}1 \to 2\\ 2 \to 3\\ 3 \to 1\end{matrix} \qquad b: \begin{matrix}1 \to 2\\ 2 \to 1\\ 3 \to 3\end{matrix}\right]$

generators: a and b

relations: $a^3 = 1$, $b^2 = 1$, $a^2 b = ba$

Let's analyze the conjugation action of $G = S_3$ on itself

$$G \ni x \longmapsto g * x = gxg^{-1} \in G \qquad \text{for } g \in G$$

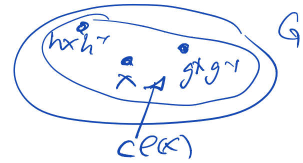
- For such type of action, the orbits are the <u>conjugacy classes</u>

$$\boxed{Cl(x) := Gx = \{y \in G : y = gxg^{-1} \text{ for some } g \in G\}}$$

conjugacy class of x inside G

orbit of x under conjugation action

[not a subgroup in general!]



- Also, the stabilizers in this case are

$$\boxed{G_x = \{g \in G : gxg^{-1} = x\} = \{g : gx = xg\} = C(x)}$$

stabilizer of $x \in G$ under conjugation action

centralizer of x in G

- Fixed point set:

$$\boxed{Fix_G(G) = G^G = \{x \in G : \underset{\substack{\updownarrow \\ gx = xg}}{gxg^{-1} = x}, \ \forall g \in G\} = Z(G)}$$

center of G (a normal subgroup!)

Back to S_3: Start by listing the conjugacy classes

- $Cl(e) = \{geg^{-1} : g \in G\} = \{e\}$
- $Cl(a) = \{a, a^2\}$

$$bab^{-1} = (ba)b = (a^2b)b = a^2b^2 = a^2$$

$\boxed{a^2 b = ba}$

$a^{-1} = a^2 \Leftarrow a^3 = 1$

$b^{-1} = b \Leftarrow b^2 = 1$

$S_3 = \left\{ \begin{matrix} e, a, a^2 \\ b, ab, a^2b \end{matrix} \right\}$

Remark [This is a complete list of conjugates of a, since $o(a) = 3$, and conjugation in a finite group preserves orders of elements]

- $Cl(b) = \{b, a^2b, ab\}$

$$aba^{-1} = aba^2 = a(ba)a = a(a^2b)a = a^3ba = ba = a^2b$$

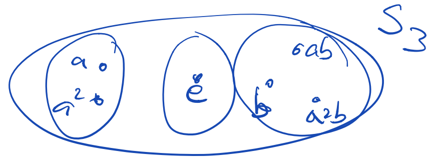
$$a^2ba^{-2} = ab \qquad \text{(exercise)}$$

To recap, there are 3 conjugacy classes in S_3:

$\{e\}$ ← order 1

$\{a, a^2\}$ ← order 3

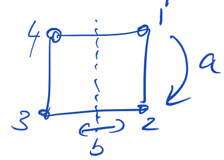
$\{b, a^2b, ab\}$ ← order 2



Note: $Z(S_3) = \{e\}$. Exercise: Compute $C(x)$, for $x \in S_3$

(3) Conjugation action on D_4 (symmetries of the square)

$$D_4 = \{ \underset{1}{e}, \underset{4}{a}, \underset{2}{a^2}, \underset{4}{a^3}, \; b, ab, a^2b, a^3b \}$$



$$\boxed{\begin{array}{l} ba = a^{-1}b \\ b^2 = a^4 = 1 \end{array}}$$

Conjugacy classes

$$\underset{1}{\{e\}}, \; \underset{4}{\{a, a^3\}}, \; \underset{2}{\{a^2\}}, \; \underset{2}{\{b, a^2b\}}, \; \underset{2}{\{ab, a^3b\}}$$

$a(ba^{-1}) = a(ab) = a^2b$ $\qquad$ $a(ab)a^{-1} = a^2(ba^{-1}) = a^2(ab) = a^3b$

Fixed point set: $Z(D_4) = \{e, a^2\}$ $\qquad$ $(ba^2 = a^{-2}b = a^2b)$

(4) Conjugacy classes of permutations

Question: How to find efficiently $Cl(\sigma) = \{\tau\sigma\tau^{-1} : \tau \in S_n\}$ for every permutation $\sigma \in S_n$?

The answer involves two steps:

(1) Write σ as a product of cycles.

(2) Show that any two permutations with the same cycle shape are conjugate.

Step 1 (Illustrate on an example $\sigma \in S_9$)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 2 & 1 & 9 & 6 & 5 & 3 & 4 & 8 \end{pmatrix} = (173)(2)(498)(56)$$

We wrote σ as a product of disjoint (commuting!) cycles

Step 2 First note: if $\sigma(i) = j$, then $(\sigma, \tau \in S_n)$

$$\tau\sigma\tau^{-1}(\tau(i)) = \tau\sigma(i) = \tau(j)$$

i.e.: $i \xrightarrow{\sigma} j \implies \tau(i) \xrightarrow{\tau\sigma\tau^{-1}} \tau(j)$

Hence, if $\sigma = (a_1\, a_2 \cdots a_k)$ — a k-cycle

$a_1 \to a_2 \to \cdots \to a_k$

then $\tau\sigma\tau^{-1} = (\tau(a_1)\ \tau(a_2) \cdots \tau(a_k))$ — again a k-cycle!

i.e.: conjugation in S_n takes k-cycles to k-cycles.
Tracing back, any two k-cycles in S_n are conjugate.

Prop Two permutations in S_n are conjugate if and only if they have the same type of cycle decomposition.

Eg:

$$\underbrace{(\cdot\cdot\cdot)}_{3}\underbrace{(\cdot\cdot)}_{2}\underbrace{(\cdot)}_{1}\underbrace{(\cdot\cdot\cdot\cdot)}_{4}\underbrace{(\cdot)}_{1}\underbrace{(\cdot\cdot\cdot\cdot\cdot)}_{5}\underbrace{(\cdot\cdot)}_{2}$$

$$\Updownarrow$$

$$\underbrace{(\cdot)}_{1}\underbrace{(\cdot\cdot\cdot\cdot\cdot)}_{5}\underbrace{(\cdot\cdot)}_{2}\underbrace{(\cdot\cdot\cdot)}_{3}\underbrace{(\cdot)}_{1}\underbrace{(\cdot\cdot\cdot\cdot)}_{4}\underbrace{(\cdot\cdot)}_{2} \quad \to 18$$

Easy example: Conjugacy classes in S_3 revisited

(1) $(\cdot)(\cdot)(\cdot) \to e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (1)(2)(3)$ 1

(2) $(\cdot)(\cdot\cdot) \to \{(1)(23),\ (2)(13),\ (3)(12)\}$ 3

(3) $(\cdot\cdot\cdot) \to \{(123),\ (132)\}$ 2

For S_n, we codify these cycle lengths as partitions of n:

$$\boxed{n = k_1 + \cdots + k_r \qquad 1 \le k_i \le n}$$

$(1 \le r \le n)$

$$\underbrace{[\underbrace{\cdot\cdot\cdot\cdot}_{k_1} | \underbrace{\cdot\cdot\cdot}_{k_2} | \cdots | \underbrace{\cdot\cdot\cdot\cdot}_{k_r}]}_{n}$$

For instance: $18 = 1 + 1 + 2 + 2 + 3 + 4 + 5$

$\boxed{S_3}$ partition		conj. classes	# el. in conj. class
111	—	$()$	1
12	—	$(12), (13), (23)$	3
3	—	$(123), (132)$	2
			6

$\boxed{S_4}$

1111	( )	1
112	(12), (13), (14), (23), (24), (34)	6
13	(123), (132), (124), (142), (134), (143), (234), (243)	8
22	(12)(34), (13)(24), (14)(23)	3
4	(1234), (1324), (1432), (1423), (1342), (1243)	6
		24

$\boxed{S_5}$

Partition	Element in conj class	Size of conj class
11111	( )	1
1112	(12)	$\binom{5}{2} = 10$
113	(123)	$\binom{5}{3} \cdot 2 = 20$
122	(12)(34)	$\binom{5}{2}\binom{3}{2}\cdot\frac{1}{2} = 15$
23	(12)(345)	$\binom{5}{2}\cdot 2 = 20$
14	(1234)	$5 \cdot 4! = 30$
5	(12345)	$1 \cdot 4! = 24$
		120